

Geometry & Topology Monographs
 Volume 3: Invitation to higher local fields
 Part II, section 6, pages 263–272

6. Φ - Γ -modules and Galois cohomology

Laurent Herr

6.0. Introduction

Let G be a profinite group and p a prime number.

Definition. A finitely generated \mathbb{Z}_p -module V endowed with a continuous G -action is called a \mathbb{Z}_p -adic representation of G . Such representations form a category denoted by $\text{Rep}_{\mathbb{Z}_p}(G)$; its subcategory $\text{Rep}_{\mathbb{F}_p}(G)$ (respectively $\text{Rep}_{p\text{-tor}}(G)$) of mod p representations (respectively p -torsion representations) consists of the V annihilated by p (respectively a power of p).

Problem. To calculate in a simple explicit way the cohomology groups $H^i(G, V)$ of the representation V .

A method to solve it for $G = G_K$ (K is a local field) is to use Fontaine's theory of Φ - Γ -modules and pass to a simpler Galois representation, paying the price of enlarging \mathbb{Z}_p to the ring of integers of a two-dimensional local field. In doing this we have to replace linear with semi-linear actions.

In this paper we give an overview of the applications of such techniques in different situations. We begin with a simple example.

6.1. The case of a field of positive characteristic

Let E be a field of characteristic p , $G = G_E$ and $\sigma: E^{\text{sep}} \rightarrow E^{\text{sep}}$, $\sigma(x) = x^p$ the absolute Frobenius map.

Definition. For $V \in \text{Rep}_{\mathbb{F}_p}(G_E)$ put $D(V) := (E^{\text{sep}} \otimes_{\mathbb{F}_p} V)^{G_E}$; σ acts on $D(V)$ by acting on E^{sep} .

Properties.

- (1) $\dim_E D(V) = \dim_{\mathbb{F}_p} V$;
- (2) the “Frobenius” map $\varphi: D(V) \rightarrow D(V)$ induced by $\sigma \otimes \text{id}_V$ satisfies:
 - a) $\varphi(\lambda x) = \sigma(\lambda)\varphi(x)$ for all $\lambda \in E$, $x \in D(V)$ (so φ is σ -semilinear);
 - b) $\varphi(D(V))$ generates $D(V)$ as an E -vector space.

Definition. A finite dimensional vector space M over E is called an *étale Φ -module* over E if there is a σ -semilinear map $\varphi: M \rightarrow M$ such that $\varphi(M)$ generates M as an E -vector space.

Étale Φ -modules form an abelian category $\Phi M_E^{\text{ét}}$ (the morphisms are the linear maps commuting with the Frobenius φ).

Theorem 1 (Fontaine, [F]). *The functor $V \rightarrow D(V)$ is an equivalence of the categories $\text{Rep}_{\mathbb{F}_p}(G_E)$ and $\Phi M_E^{\text{ét}}$.*

We see immediately that $H^0(G_E, V) = V^{G_E} \simeq D(V)^\varphi$.

So in order to obtain an explicit description of the Galois cohomology of mod p representations of G_E , we should try to derive in a simple manner the functor associating to an étale Φ -module the group of points fixed under φ . This is indeed a much simpler problem because there is only one operator acting.

For $(M, \varphi) \in \Phi M_E^{\text{ét}}$ define the following complex of abelian groups:

$$C_1(M) : \quad 0 \rightarrow M \xrightarrow{\varphi-1} M \rightarrow 0$$

(M stands at degree 0 and 1).

This is a functorial construction, so by taking the cohomology of the complex, we obtain a cohomological functor $(\mathcal{H}^i := H^i \circ C_1)_{i \in \mathbb{N}}$ from $\Phi M_E^{\text{ét}}$ to the category of abelian groups.

Theorem 2. *The cohomological functor $(\mathcal{H}^i \circ D)_{i \in \mathbb{N}}$ can be identified with the Galois cohomology functor $(H^i(G_E, \cdot))_{i \in \mathbb{N}}$ for the category $\text{Rep}_{\mathbb{F}_p}(G_E)$. So, if $M = D(V)$ then $\mathcal{H}^i(M)$ provides a simple explicit description of $H^i(G_E, V)$.*

Proof of Theorem 2. We need to check that the cohomological functor $(\mathcal{H}^i)_{i \in \mathbb{N}}$ is universal; therefore it suffices to verify that for every $i \geq 1$ the functor \mathcal{H}^i is effaceable: this means that for every $(M, \varphi_M) \in \Phi M_E^{\text{ét}}$ and every $x \in \mathcal{H}^i(M)$ there exists an embedding u of (M, φ_M) in $(N, \varphi_N) \in \Phi M_E^{\text{ét}}$ such that $\mathcal{H}^i(u)(x)$ is zero in $\mathcal{H}^i(N)$. But this is easy: it is trivial for $i \geq 2$; for $i = 1$ choose an element m belonging to the class $x \in M/(\varphi - 1)(M)$, put $N := M \oplus Et$ and extend φ_M to the σ -semi-linear map φ_N determined by $\varphi_N(t) := t + m$. \square

6.2. Φ - Γ -modules and \mathbb{Z}_p -adic representations

Definition. Recall that a Cohen ring is an absolutely unramified complete discrete valuation ring of mixed characteristic $(0, p > 0)$, so its maximal ideal is generated by p .

We describe a general formalism, explained by Fontaine in [F], which lifts the equivalence of categories of Theorem 1 in characteristic 0 and relates the \mathbb{Z}_p -adic representations of G to a category of modules over a Cohen ring, endowed with a “Frobenius” map and a group action.

Let R be an algebraically closed complete valuation (of rank 1) field of characteristic p and let H be a normal closed subgroup of G . Suppose that G acts continuously on R by ring automorphisms. Then $F := R^H$ is a perfect closed subfield of R .

For every integer $n \geq 1$, the ring $W_n(R)$ of Witt vectors of length n is endowed with the product of the topology on R defined by the valuation and then $W(R)$ with the inverse limit topology. Then the componentwise action of the group G is continuous and commutes with the natural Frobenius σ on $W(R)$. We also have $W(R)^H = W(F)$.

Let E be a closed subfield of F such that F is the completion of the p -radical closure of E in R . Suppose there exists a Cohen subring \mathcal{O}_E of $W(R)$ with residue field E and which is stable under the actions of σ and of G . Denote by $\mathcal{O}_{\widehat{E}_{\text{ur}}}$ the completion of the integral closure of \mathcal{O}_E in $W(R)$: it is a Cohen ring which is stable by σ and G , its residue field is the separable closure of E in R and $(\mathcal{O}_{\widehat{E}_{\text{ur}}})^H = \mathcal{O}_E$.

The natural map from H to G_E is an isomorphism if and only if the action of H on R induces an isomorphism from H to G_F . We suppose that this is the case.

Definition. Let Γ be the quotient group G/H . An étale Φ - Γ -module over \mathcal{O}_E is a finitely generated \mathcal{O}_E -module endowed with a σ -semi-linear Frobenius map $\varphi: M \rightarrow M$ and a continuous Γ -semi-linear action of Γ commuting with φ such that the image of φ generates the module M .

Étale Φ - Γ -modules over \mathcal{O}_E form an abelian category $\Phi\Gamma M_{\mathcal{O}_E}^{\text{ét}}$ (the morphisms are the linear maps commuting with φ). There is a tensor product of Φ - Γ -modules, the natural one. For two objects M and N of $\Phi\Gamma M_{\mathcal{O}_E}^{\text{ét}}$ the \mathcal{O}_E -module $\text{Hom}_{\mathcal{O}_E}(M, N)$ can be endowed with an étale Φ - Γ -module structure (see [F]).

For every \mathbb{Z}_p -adic representation V of G , let $D_H(V)$ be the \mathcal{O}_E -module $(\mathcal{O}_{\widehat{E}_{\text{ur}}} \otimes_{\mathbb{Z}_p} V)^H$. It is naturally an étale Φ - Γ -module, with φ induced by the map $\sigma \otimes \text{id}_V$ and Γ acting on both sides of the tensor product. From Theorem 2 one deduces the following fundamental result:

Theorem 3 (Fontaine, [F]). *The functor $V \rightarrow D_H(V)$ is an equivalence of the categories $\text{Rep}_{\mathbb{Z}_p}(G)$ and $\Phi\Gamma M_{\mathcal{O}_E}^{\text{ét}}$.*

Remark. If E is a field of positive characteristic, \mathcal{O}_E is a Cohen ring with residue field E endowed with a Frobenius σ , then we can easily extend the results of the whole subsection 6.1 to \mathbb{Z}_p -adic representations of G by using Theorem 3 for $G = G_E$ and $H = \{1\}$.

6.3. A brief survey of the theory of the field of norms

For the details we refer to [W], [FV] or [F].

Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic p . Put $G = G_K = \text{Gal}(K^{\text{sep}}/K)$.

Let \mathbb{C} be the completion of K^{sep} , denote the extension of the discrete valuation v_K of K to \mathbb{C} by v_K . Let $R^* = \varprojlim \mathbb{C}_n^*$ where $\mathbb{C}_n = \mathbb{C}$ and the morphism from \mathbb{C}_{n+1} to \mathbb{C}_n is raising to the p th power. Put $R := R^* \cup \{0\}$ and define $v_R((x_n)) = v_K(x_0)$. For $(x_n), (y_n) \in R$ define

$$(x_n) + (y_n) = (z_n) \quad \text{where } z_n = \lim_m (x_{n+m} + y_{n+m})^{p^m}.$$

Then R is an algebraically closed field of characteristic p complete with respect to v_R (cf. [W]). Its residue field is isomorphic to the algebraic closure of k and there is a natural continuous action of G on R . (Note that Fontaine denotes this field by $\text{Fr } R$ in [F]).

Let L be a Galois extension of K in K^{sep} . Recall that one can always define the ramification filtration on $\text{Gal}(L/K)$ in the upper numbering. Roughly speaking, L/K is an arithmetically profinite extension if one can define the lower ramification subgroups of G so that the classical relations between the two filtrations for finite extensions are preserved. This is in particular possible if $\text{Gal}(L/K)$ is a p -adic Lie group with finite residue field extension.

The field R contains in a natural way the field of norms $N(L/K)$ of every arithmetically profinite extension L of K and the restriction of v to $N(L/K)$ is a discrete valuation. The residue field of $N(L/K)$ is isomorphic to that of L and $N(L/K)$ is stable under the action of G . The construction is functorial: if L' is a finite extension of L contained in K^{sep} , then L'/K is still arithmetically profinite and $N(L'/K)$ is a separable extension of $N(L/K)$. The direct limit of the fields $N(L'/K)$ where L' goes through all the finite extensions of L contained in K^{sep} is the separable closure E^{sep} of $E = N(L/K)$. It is stable under the action of G and the subgroup G_L identifies with G_E . The field E^{sep} is dense in R .

Fontaine described how to lift these constructions in characteristic 0 when L is the cyclotomic \mathbb{Z}_p -extension K_∞ of K . Consider the ring of Witt vectors $W(R)$ endowed with the Frobenius map σ and the natural componentwise action of G . Define the topology of $W(R)$ as the product of the topology defined by the valuation on R . Then one can construct a Cohen ring $\mathcal{O}_{\widehat{E}_{\text{ur}}}$ with residue field E^{sep} ($E = N(L/K)$) such that:

- (i) $\mathcal{O}_{\widehat{\mathcal{E}}_{\text{ur}}}$ is stable by σ and the action of G ,
- (ii) for every finite extension L of K_∞ the ring $(\mathcal{O}_{\widehat{\mathcal{E}}_{\text{ur}}})^{G_L}$ is a Cohen ring with residue field E .

Denote by $\mathcal{O}_{\mathcal{E}(K)}$ the ring $(\mathcal{O}_{\widehat{\mathcal{E}}_{\text{ur}}})^{G_{K_\infty}}$. It is stable by σ and the quotient $\Gamma = G/G_{K_\infty}$ acts continuously on $\mathcal{O}_{\mathcal{E}(K)}$ with respect to the induced topology. Fix a topological generator γ of Γ : it is a continuous ring automorphism commuting with σ . The fraction field of $\mathcal{O}_{\mathcal{E}(K)}$ is a two-dimensional standard local field (as defined in section 1 of Part I). If π is a lifting of a prime element of $N(K_\infty/K)$ in $\mathcal{O}_{\mathcal{E}(K)}$ then the elements of $\mathcal{O}_{\mathcal{E}(K)}$ are the series $\sum_{i \in \mathbb{Z}} a_i \pi^i$, where the coefficients a_i are in $W(k_{K_\infty})$ and converge p -adically to 0 when $i \rightarrow -\infty$.

6.4. Application of \mathbb{Z}_p -adic representations of G to the Galois cohomology

If we put together Fontaine's construction and the general formalism of subsection 6.2 we obtain the following important result:

Theorem 3' (Fontaine, [F]). *The functor $V \rightarrow D(V) := (\mathcal{O}_{\widehat{\mathcal{E}}_{\text{ur}}} \otimes_{\mathbb{Z}_p} V)^{G_{K_\infty}}$ defines an equivalence of the categories $\text{Rep}_{\mathbb{Z}_p}(G)$ and $\Phi\Gamma M_{\mathcal{O}_{\mathcal{E}(K)}}^{\text{ét}}$.*

Since for every \mathbb{Z}_p -adic representation of G we have $H^0(G, V) = V^G \simeq D(V)^\varphi$, we want now, as in paragraph 6.1, compute explicitly the cohomology of the representation using the Φ - Γ -module associated to V .

For every étale Φ - Γ -module (M, φ) define the following complex of abelian groups:

$$C_2(M) : \quad 0 \rightarrow M \xrightarrow{\alpha} M \oplus M \xrightarrow{\beta} M \rightarrow 0$$

where M stands at degree 0 and 2,

$$\alpha(x) = ((\varphi - 1)x, (\gamma - 1)x), \quad \beta((y, z)) = (\gamma - 1)y - (\varphi - 1)z.$$

By functoriality, we obtain a cohomological functor $(\mathcal{H}^i := H^i \circ C_2)_{i \in \mathbb{N}}$ from $\Phi\Gamma M_{\mathcal{O}_{\mathcal{E}(K)}}^{\text{ét}}$ to the category of abelian groups.

Theorem 4 (Herr, [H]). *The cohomological functor $(\mathcal{H}^i \circ D)_{i \in \mathbb{N}}$ can be identified with the Galois cohomology functor $(H^i(G, \cdot))_{i \in \mathbb{N}}$ for the category $\text{Rep}_{p\text{-tor}}(G)$. So, if $M = D(V)$ then $\mathcal{H}^i(M)$ provides a simple explicit description of $H^i(G, V)$ in the p -torsion case.*

Idea of the proof of Theorem 4. We have to check that for every $i \geq 1$ the functor \mathcal{H}^i is effaceable. For every p -torsion object $(M, \varphi_M) \in \Phi\Gamma M_{\mathcal{O}_{\mathcal{E}(K)}}^{\text{ét}}$ and every $x \in \mathcal{H}^i(M)$ we construct an explicit embedding u of (M, φ_M) in a certain $(N, \varphi_N) \in \Phi\Gamma M_{\mathcal{O}_{\mathcal{E}(K)}}^{\text{ét}}$

such that $\mathcal{H}^i(u)(x)$ is zero in $\mathcal{H}^i(N)$. For details see [H]. The key point is of topological nature: we prove, following an idea of Fontaine in [F], that there exists an open neighbourhood of 0 in M on which $(\varphi - 1)$ is bijective and use then the continuity of the action of Γ . \square

As an application of theorem 4 we can prove the following result (due to Tate):

Theorem 5. *Assume that k_K is finite and let V be in $\text{Rep}_{p\text{-tor}}(G)$. Without using class field theory the previous theorem implies that $H^i(G, V)$ are finite, $H^i(G, V) = 0$ for $i \geq 3$ and*

$$\sum_{i=0}^2 l(H^i(G, V)) = -|K: \mathbb{Q}_p| l(V),$$

where $l(\cdot)$ denotes the length over \mathbb{Z}_p .

See [H].

Remark. Because the finiteness results imply that the Mittag–Leffler conditions are satisfied, it is possible to generalize the explicit construction of the cohomology and to prove analogous results for \mathbb{Z}_p (or \mathbb{Q}_p)-adic representations by passing to the inverse limits.

6.5. A new approach to local class field theory

The results of the preceding paragraph allow us to prove without using class field theory the following:

Theorem 6 (Tate’s local duality). *Let V be in $\text{Rep}_{p\text{-tor}}(G)$ and $n \in \mathbb{N}$ such that $p^n V = 0$. Put $V^*(1) := \text{Hom}(V, \mu_{p^n})$. Then there is a canonical isomorphism from $H^2(G, \mu_{p^n})$ to \mathbb{Z}/p^n and the cup product*

$$H^i(G, V) \times H^{2-i}(G, V^*(1)) \xrightarrow{\cup} H^2(G, \mu_{p^n}) \simeq \mathbb{Z}/p^n$$

is a perfect pairing.

It is well known that a proof of the local duality theorem of Tate without using class field theory gives a construction of the reciprocity map. For every $n \geq 1$ we have by duality a functorial isomorphism between the finite groups $\text{Hom}(G, \mathbb{Z}/p^n) = H^1(G, \mathbb{Z}/p^n)$ and $H^1(G, \mu_{p^n})$ which is isomorphic to $K^*/(K^*)^{p^n}$ by Kummer theory. Taking the inverse limits gives us the p -part of the reciprocity map, the most difficult part.

Sketch of the proof of Theorem 6. ([H2]).

a) Introduction of differentials:

Let us denote by Ω_c^1 the $\mathcal{O}_{\mathcal{E}(K)}$ -module of continuous differential forms of $\mathcal{O}_{\mathcal{E}}$ over $W(k_{K_\infty})$. If π is a fixed lifting of a prime element of $E(K_\infty/K)$ in $\mathcal{O}_{\mathcal{E}(K)}$, then this module is free and generated by $d\pi$. Define the residue map from Ω_c^1 to $W(k_{K_\infty})$ by $\text{res}(\sum_{i \in \mathbb{Z}} a_i \pi^i d\pi) := a_{-1}$; it is independent of the choice of π .

b) Calculation of some Φ - Γ -modules:

The $\mathcal{O}_{\mathcal{E}(K)}$ -module Ω_c^1 is endowed with an étale Φ - Γ -module structure by the following formulas: for every $\lambda \in \mathcal{O}_{\mathcal{E}(K)}$ we put:

$$p\varphi(\lambda d\pi) = \sigma(\lambda)d(\sigma(\pi)) \quad , \quad \gamma(\lambda d\pi) = \gamma(\lambda)d(\gamma(\pi)).$$

The fundamental fact is that there is a natural isomorphism of Φ - Γ -modules over $\mathcal{O}_{\mathcal{E}(K)}$ between $D(\mu_{p^n})$ and the reduction $\Omega_{c,n}^1$ of Ω_c^1 modulo p^n .

The étale Φ - Γ -module associated to the representation $V^*(1)$ is

$\widetilde{M} := \text{Hom}(M, \Omega_{c,n}^1)$, where $M = D(V)$. By composing the residue with the trace we obtain a surjective and continuous map Tr_n from M to \mathbb{Z}/p^n . For every $f \in \widetilde{M}$, the map $\text{Tr}_n \circ f$ is an element of the group M^\vee of continuous group homomorphisms from M to \mathbb{Z}/p^n . This gives in fact a group isomorphism from \widetilde{M} to M^\vee and we can therefore transfer the Φ - Γ -module structure from \widetilde{M} to M^\vee . But, since k is finite, M is locally compact and M^\vee is in fact the Pontryagin dual of M .

c) Pontryagin duality implies local duality:

We simply dualize the complex $C_2(M)$ using Pontryagin duality (all arrows are strict morphisms in the category of topological groups) and obtain a complex:

$$C_2(M)^\vee : \quad 0 \rightarrow M^\vee \xrightarrow{\beta^\vee} M^\vee \oplus M^\vee \xrightarrow{\alpha^\vee} M^\vee \rightarrow 0,$$

where the two M^\vee are in degrees 0 and 2. Since we can construct an explicit quasi-isomorphism between $C_2(M^\vee)$ and $C_2(M)^\vee$, we easily obtain a duality between $\mathcal{H}^i(M)$ and $\mathcal{H}^{2-i}(M^\vee)$ for every $i \in \{0, 1, 2\}$.

d) The canonical isomorphism from $\mathcal{H}^2(\Omega_{c,n}^1)$ to \mathbb{Z}/p^n :

The map Tr_n from $\Omega_{c,n}^1$ to \mathbb{Z}/p^n factors through the group $\mathcal{H}^2(\Omega_{c,n}^1)$ and this gives an isomorphism. But it is not canonical! In fact the construction of the complex $C_2(M)$ depends on the choice of γ . Fortunately, if we take another γ , we get a quasi-isomorphic complex and if we normalize the map Tr_n by multiplying it by the unit $-p^{v_p(\log \chi(\gamma))} / \log \chi(\gamma)$ of \mathbb{Z}_p , where \log is the p -adic logarithm, χ the cyclotomic character and $v_p = v_{\mathbb{Q}_p}$, then everything is compatible with the change of γ .

e) The duality is given by the cup product:

We can construct explicit formulas for the cup product:

$$\mathcal{H}^i(M) \times \mathcal{H}^{2-i}(M^\vee) \xrightarrow{\cup} \mathcal{H}^2(\Omega_{c,n}^1)$$

associated with the cohomological functor $(\mathcal{H}^i)_{i \in \mathbb{N}}$ and we compose them with the preceding normalized isomorphism from $\mathcal{H}^2(\Omega_{c,n}^1)$ to \mathbb{Z}/p^n . Since everything is explicit, we can compare with the pairing obtained in c) and verify that it is the same up to a unit of \mathbb{Z}_p . \square

Remark. Benois, using the previous theorem, deduced an explicit formula of Coleman's type for the Hilbert symbol and proved Perrin-Riou's formula for crystalline representations ([B]).

6.6. Explicit formulas for the generalized Hilbert symbol on formal groups

Let K_0 be the fraction field of the ring W_0 of Witt vectors with coefficients in a finite field of characteristic $p > 2$ and \mathcal{F} a commutative formal group of finite height h defined over W_0 .

For every integer $n \geq 1$, denote by $\mathcal{F}[p^n]$ the p^n -torsion points in $\mathcal{F}(\mathcal{M}_C)$, where \mathcal{M}_C is the maximal ideal of the completion C of an algebraic closure of K_0 . The group $\mathcal{F}[p^n]$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^h$.

Let K be a finite extension of K_0 contained in K^{sep} and assume that the points of $\mathcal{F}[p^n]$ are defined over K . We then have a bilinear pairing:

$$(\ , \]_{\mathcal{F},n}: G_K^{\text{ab}} \times \mathcal{F}(\mathcal{M}_K) \rightarrow \mathcal{F}[p^n]$$

(see section 8 of Part I).

When the field K contains a primitive p^n th root of unity ζ_{p^n} , Abrashkin gives an explicit description for this pairing generalizing the classical Brückner–Vostokov formula for the Hilbert symbol ([A]). In his paper he notices that the formula makes sense even if K does not contain ζ_{p^n} and he asks whether it holds without this assumption. In a recent unpublished work, Benois proves that this is true.

Suppose for simplicity that K contains only ζ_p . Abrashkin considers in his paper the extension $\tilde{K} := K(\pi^{p^{-r}})$, $r \geq 1$, where π is a fixed prime element of K . It is not a Galois extension of K but is arithmetically profinite, so by [W] one can consider the field of norms for it. In order not to lose information given by the roots of unity of order a power of p , Benois uses the composite Galois extension $L := K_\infty \tilde{K}/K$ which is arithmetically profinite. There are several problems with the field of norms $N(L/K)$, especially it is not clear that one can lift it in characteristic 0 with its Galois action. So, Benois simply considers the completion F of the p -radical closure of $E = N(L/K)$ and its separable closure F^{sep} in R . If we apply what was explained in subsection 6.2 for $\Gamma = \text{Gal}(L/K)$, we get:

Theorem 7. *The functor $V \rightarrow D(V) := (W(F^{\text{sep}}) \otimes_{\mathbb{Z}_p} V)^{G_L}$ defines an equivalence of the categories $\text{Rep}_{\mathbb{Z}_p}(G)$ and $\Phi\Gamma M_{W(F)}^{\text{ét}}$.*

Choose a topological generator γ' of $\text{Gal}(L/K_\infty)$ and lift γ to an element of $\text{Gal}(L/\tilde{K})$. Then Γ is topologically generated by γ and γ' , with the relation $\gamma\gamma' = (\gamma')^a\gamma$, where $a = \chi(\gamma)$ (χ is the cyclotomic character). For $(M, \varphi) \in \Phi\Gamma M_{W(F)}^{\text{ét}}$ the continuous action of $\text{Gal}(L/K_\infty)$ on M makes it a module over the Iwasawa algebra $\mathbb{Z}_p[[\gamma' - 1]]$. So we can define the following complex of abelian groups:

$$C_3(M) : \quad 0 \rightarrow M_0 \xrightarrow{\alpha \mapsto A_0 \alpha} M_1 \xrightarrow{\alpha \mapsto A_1 \alpha} M_2 \xrightarrow{\alpha \mapsto A_2 \alpha} M_3 \rightarrow 0$$

where M_0 is in degree 0, $M_0 = M_3 = M$, $M_1 = M_2 = M^3$,

$$A_0 = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \\ \gamma' - 1 \end{pmatrix}, A_1 = \begin{pmatrix} \gamma - 1 & 1 - \varphi & 0 \\ \gamma' - 1 & 0 & 1 - \varphi \\ 0 & \gamma'^a - 1 & \delta - \gamma \end{pmatrix}, A_2 = ((\gamma')^a - 1 \quad \delta - \gamma \quad \varphi - 1)$$

and $\delta = ((\gamma')^a - 1)(\gamma' - 1)^{-1} \in \mathbb{Z}_p[[\gamma' - 1]]$.

As usual, by taking the cohomology of this complex, one defines a cohomological functor $(\mathcal{H}^i)_{i \in \mathbb{N}}$ from $\Phi\Gamma M_{W(F)}^{\text{ét}}$ in the category of abelian groups. Benois proves only that the cohomology of a p -torsion representation V of G injects in the groups $\mathcal{H}^i(D(V))$ which is enough to get the explicit formula. But in fact a stronger statement is true:

Theorem 8. *The cohomological functor $(\mathcal{H}^i \circ D)_{i \in \mathbb{N}}$ can be identified with the Galois cohomology functor $(H^i(G, \cdot))_{i \in \mathbb{N}}$ for the category $\text{Rep}_{p\text{-tor}}(G)$.*

Idea of the proof. Use the same method as in the proof of Theorem 4. It is only more technically complicated because of the structure of Γ . \square

Finally, one can explicitly construct the cup products in terms of the groups \mathcal{H}^i and, as in [B], Benois uses them to calculate the Hilbert symbol.

Remark. Analogous constructions (equivalence of category, explicit construction of the cohomology by a complex) seem to work for higher dimensional local fields. In particular, in the two-dimensional case, the formalism is similar to that of this paragraph; the group Γ acting on the Φ - Γ -modules has the same structure as here and thus the complex is of the same form. This work is still in progress.

References

- [A] V. A. Abrashkin, Explicit formulae for the Hilbert symbol of a formal group over the Witt vectors, *Izv. RAN Math.* 61(1997), 463–515.
- [B] D. Benois, On Iwasawa theory of crystalline representations, *Duke Math. J.* 104 (2) (2000), 211–267.

- [FV] I. Fesenko and S. Vostokov, Local Fields and Their Extensions, Trans. of Math. Monographs, 121, A.M.S., 1993.
- [F] J.-M. Fontaine, Représentations p -adiques des corps locaux, The Grothendieck Festschrift 2, Birkhäuser, 1994, 59–111.
- [H1] L. Herr, Sur la cohomologie galoisienne des corps p -adiques, Bull. de la Soc. Math. de France, 126(1998), 563–600.
- [H2] L. Herr, Une approche nouvelle de la dualité locale de Tate, to appear in Math. Annalen.
- [W] J.-P. Wintenberger, Le corps des normes de certaines extensions infinies de corps locaux; applications, Ann. Sci. E.N.S. 16(1983), 59–89.

*Université Bordeaux I, Laboratoire A2X,
351 Cours de la Libération, 33405 Talence, France
E-mail: herr@math.u-bordeaux.fr*